ECEN/MAE 5513

Homework #7

1. The random variable X has a density function $f_X(\lambda)$ defined as

$$f_X(\lambda) = \begin{cases} \dfrac{\theta}{\lambda^2} & \lambda \geq \theta \\ 0 & \lambda < \theta \end{cases}$$

Take one sample of the random variable - X_1. **a)** Find and sketch the likelihood function. **b)** Find the maximum likelihood estimate for θ.

2. Using the sample of the random variable from problem 1, assume that θ is a random variable with the following density function.

$$f_\theta(\mu) = \begin{cases} \dfrac{1}{\mu^2} & \mu \geq 1 \\ 0 & \mu < 1 \end{cases}$$

a) Sketch $f_{\theta|X_1}(\mu|\lambda)$, **b)** Find the Bayesian maximum aposteriore estimate $\hat{\theta}_{MAP}$.

3. Consider the measurement (Z) of a signal (X) in noise (V):

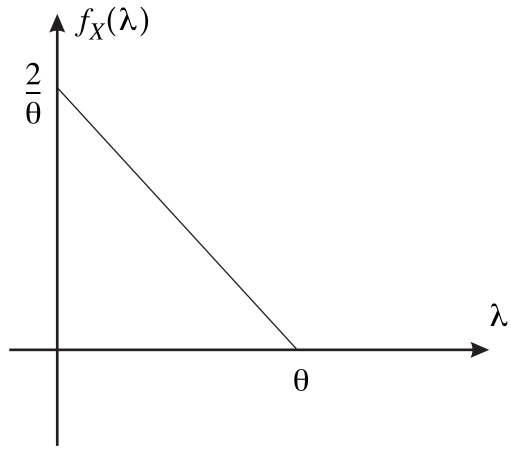
$$Z = X + V$$

$$f_V(\mu) = \begin{cases} 1 & 0 \leq \mu \leq 1 \\ 0 & else \end{cases}$$

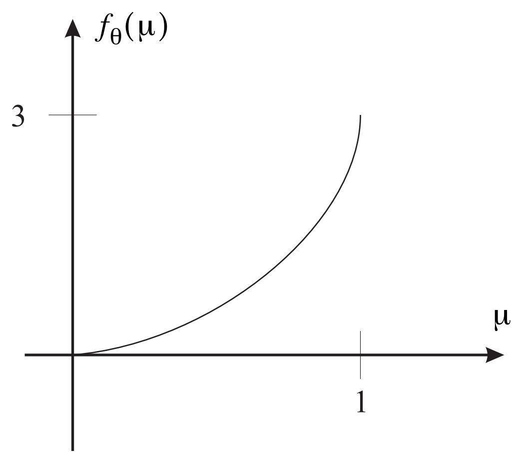
$$f_X(\lambda) = \begin{cases} 1 & 1 \leq \lambda \leq 2 \\ 0 & else \end{cases}$$

a) Sketch $f_V(\mu)$ and $f_X(\lambda)$, **b)** Sketch $f_{Z|X}(\gamma|\lambda)$ vs. γ for a fixed λ. c) Sketch $f_{Z|X}(\gamma|\lambda)$ vs. λ for $\gamma = 1.5$. d) Sketch $f_{X|Z}(\lambda|1.5)$ vs. λ. If the measurement $Z = 1.5$, find the Bayesian minimum mean square error estimate of the signal: $\hat{X}_{MS} = E[X|Z = 1.5]$.

4. The random variable X has the density function shown below. One sample, X_1, of the random variable is taken. Find the maximum likelihood estimate of the unknown constant θ. **Show each step clearly**.



5. Consider again the random variable X from problem 4. Assume now that θ is a random variable, with prior density given below. Find the MAP estimate of θ, based on the single sample X_1. **Show each step clearly.**



$$f_\theta(\mu) = \begin{cases} 3\mu^2 & 0 < \mu < 1 \\ 0 & else \end{cases}$$

6. Consider the measurement (Z) of a signal (X) in noise (V):

$$Z = X + V,$$

where V is Gaussian noise with zero mean and unity variance. The signal is transmitted N times, and N independent measurements $Z_1, Z_2, \ldots, Z_N$ are received.

   i. Find the joint density function for the measurements, $f_{Z_1, Z_2, \ldots, Z_N}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

   ii. Find the likelihood function $L(X)$.

   iii. Find the log likelihood function $l(X) = \ln L(X)$.

   iv. Find the maximum likelihood estimate $\hat{x}_{ML}$.

7. Assume now that the transmitted signal X from problem 6 is a Gaussian random variable with mean X_0 and variance σ^2. Assume that the same measurements $Z_1, Z_2, \ldots, Z_N$ are received.

   i. Find the Bayesian maximum aposteriore estimate $\hat{x}_{MAP}$.

   ii. What does $\hat{x}_{MAP}$ approach as $\sigma^2 \to 0$?

   iii. What does $\hat{x}_{MAP}$ approach as $\sigma^2 \to \infty$?

   iv. What does $\hat{x}_{MAP}$ approach as $N \to \infty$?